The exact moments of a ratio of quadratic forms in normal variables

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Published in:
Annales d'Économie et de Statistique

Publication date:
1986

The exact moments of a ratio of quadratic forms in normal variables

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ABSTRACT. – The exact moments of $x'Ax/x'Bx$ are obtained, where x is a normally distributed vector with some mean (possibly nonzero) and positive definite covariance matrix, A is symmetric and B positive semidefinite. These moments appear as simple integrals which can be evaluated numerically in a straightforward manner. In addition, the precise conditions for the existence of the moments are found. Some related results are also reported.

Les moments exacts d'un rapport de formes quadratiques de variables normales

RÉSUMÉ. – On obtient les moments exacts de $x'Ax/x'Bx$ où x est un vecteur normal de moyenne quelconque de matrice de covariance définie positive, A est symétrique et B semi-définie positive. Ces moments sont des intégrales simples qui peuvent être evaluées numériquement de façon simple. De plus des conditions précises pour l'existence de ces moments sont fournies. D'autres résultats sont également présentés.

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1 Introduction

Econometric or time-series estimators often take the form of ratios of quadratic forms. A very simple example is provided by the first-order autoregressive process

$$y_t = \beta y_{t-1} + u_t \qquad (t=2, \ldots, T)$$

where $|\beta| < 1$ and the u_t are i. i. d. $N(0, \sigma^2)$. Given a series of T observations, the least-squares estimator of β is

$$\hat{\beta} = \left[\sum_{t=2}^{T} y_t y_{t-1}\right] \Big/ \left[\sum_{t=2}^{T} y_{t-1}^2\right]$$

which is a ratio of two quadratic forms in $y = (y_1, \ldots, y_T)'$.

The purpose of this paper is to find the exact moments of such estimators, that is to say, estimators that can be written

$$x'Ax/x'Bx$$

where A is symmetric, B positive semidefinite and x is normally distributed with some mean (possibly nonzero) and positive definite covariance matrix. The procedure to obtain such moments was, in principle, given by SAWA [1978], but the evaluation of the moments is complicated by the fact that s-fold differentiation under the integral sign has to be performed in order to obtain the sth moment. As a result of this difficulty we know of only a couple of papers (DEGOOYER [1980], ALI [1984]) where third and fourth moments have been considered, and then only in the central case with zero mean; the evaluation of fifth and higher moments has never been attempted to our knowledge. In the present paper this problem is solved, thus opening the way to evaluate the exact moments in the central and the non-central case to an arbitrary order.

The literature on quadratic forms is vast; a good survey can be found in Chapters 28 and 29 of JOHNSON and KOTZ [1969-1970]. Regarding the expectation of products of quadratic forms we mention papers by LANCASTER [1954], KUMAR [1973], SRIVASTAVA and TIWARI [1976], MAGNUS [1978, 1979] and DON [1979]. Regarding the *distribution* (as opposed to the moments) of quadratic forms and ratios of quadratic forms we mention JOHN GURLAND'S influential papers [1948, 1953, 1955, 1956-1957], IMHOF'S [1961] important contribution, and papers by TANEJA [1976], GIDEON and GURLAND [1976], KHURI and GOOD [1977] and WALKER [1979]. For the numerical aspects the reader should consult LUGANNINI and RICE [1984]. Special cases of the *moments* of ratios of quadratic forms were recently discussed by DWIVEDI and CHAUBEY [1981] and CHAUBEY and NUR ENAYET TALUKDER [1983]. MORAN [1948] and DUFOUR and ROY [1985] gave closed-form expressions for the first and second moments of sample autocorrelations under randomness (besides results for non-normal distributions).

Finally, econometricians have been typically interested in the moments of (least-squares) estimators (SAWA [1978], HOQUE and PETERS [1986]), the distribution of forecasts arising from a first-order autoregression (PHILLIPS [1979]), and the moments of forecasts in distributed lag models (HOQUE [1985]).

The main result of the paper is stated and proved in section 5. In sections 2-4 we give five lemmas, of independent interest, which are used in the proof of the theorem in section 5. Section 6 deals with the existence of the moments and section 7 contains some remarks on computational aspects and concludes the paper.

2 Moments, Cumulants and Quadratic Forms

If x is a random variable whose sth moment $\alpha_s = \mathrm{E}\, x^s$ exists, then its characteristic function $\varphi(t)$ can be expanded in a neighbourhood of $t=0$ as follows:

$$\varphi(t) = 1 + \sum_{h=1}^{s} \frac{(it)^h}{h!} \alpha_h + o(t^s) \tag{1}$$

where

$$\lim_{t \to 0} \frac{o(t^s)}{t^s} = 0.$$

If $\varphi(t)$ can be expanded as in (1), then $\log \varphi(t)$ may be expanded as

$$\log \varphi(t) = \sum_{h=1}^{s} \frac{(it)^h}{h!} \kappa_h + o(t^s). \tag{2}$$

The quantities κ_h are called the *cumulants* of the distribution of x. The following lemma, which is a simplified version of a result given by KENDALL and STUART ([1977], section 3.14, p. 70), shows that α_s is a polynomial in the cumulants $\kappa_1, \ldots, \kappa_s$, and gives the coefficients of the polynomial.

LEMMA 1 : If x is a random variable whose sth moment α_s exists, then its first s cumulants $\kappa_1, \ldots, \kappa_s$ exist also, and α_s can be expressed as

$$\alpha_s = \sum_{v} \frac{s!}{n_1!\, n_2! \ldots !\, n_s!} \left[\frac{\kappa_1}{1!}\right]^{n_1} \left[\frac{\kappa_2}{2!}\right]^{n_2} \cdots \left[\frac{\kappa_s}{s!}\right]^{n_s} \tag{3}$$

where the summation is over all $1 \times s$ vectors $v = (n_1, n_2, \ldots, n_s)$ whose components n_j are nonnegative integers satisfying

$$n_1 + 2n_2 + \ldots + sn_s = s. \tag{4}$$

Proof: Taking exponentials on both sides of (2) gives

$$\varphi(t)=\exp\left[\sum_{r=1}^{s}\frac{(it)^r}{r!}\kappa_r\right]\exp(o(t^s))$$

$$=\exp(o(t^s))\prod_{r=1}^{s}\exp\left[\frac{(it)^r}{r!}\kappa_r\right]$$

$$=(1+o(t^s))\prod_{r=1}^{s}\left\{\sum_{j=0}^{s}\left[\frac{(it)^{rj}(\kappa_r/r!)^j}{j!}\right]+o(t^s)\right\}$$

$$=\prod_{r=1}^{s}\sum_{j=0}^{s}\left[\frac{(\kappa_r/r!)^j}{j!}(it)^{rj}\right]+o(t^s).$$

If, for $r=1, 2, \ldots, s$, we now pick out the n_r-th term in the r-th sum such that, when multiplied together, these terms give a power of t^s, we find, upon comparison with (1),

$$\frac{\alpha_s}{s!}=\sum_{v}\frac{(\kappa_1/1!)^{n_1}}{n_1!}\cdot\frac{(\kappa_2/2!)^{n_2}}{n_2!}\cdot\ldots\cdot\frac{(\kappa_s/s!)^{n_s}}{n_s!}$$

and the result follows. □

In order to implement Lemma 1 for given value of s we must find the set of $1\times s$ vectors $(n_1, n_2, \ldots, n_s)$ satisfying (4). These are easily calculated. For $s=1, 2, 3, 4$ we have

$s=1: n_1=1$

$s=2: (n_1, n_2)=(2, 0), (0,1)$

$s=3: (n_1, n_2, n_3)=(3, 0, 0), (1, 1, 0), (0, 0, 1)$

$s=4: (n_1, n_2, n_3, n_4)=(4, 0, 0, 0), (2, 1, 0, 0), (1, 0, 1, 0), (0, 2, 0, 0), (0, 0, 0, 1)$

and hence the first four moments in terms of the cumulants are

$$\alpha_1=\kappa_1$$

$$\alpha_2=\kappa_1^2+\kappa_2$$

$$\alpha_3=\kappa_1^3+3\kappa_1\kappa_2+\kappa_3$$

$$\alpha_4=\kappa_1^4+6\kappa_1^2\kappa_2+4\kappa_1\kappa_3+3\kappa_2^2+\kappa_4.$$

For computational (and theoretical) purposes it is useful to know for given s *how many* vectors $(n_1, n_2, \ldots, n_s)$ there are satisfying (4). We shall address this problem in some more detail in section 7.

Our next task is to calculate the cumulants of the distribution of a quadratic form in normal variables.

LEMMA 2 : Let x be a normally distributed $n\times 1$ vector with mean μ and positive definite covariance matrix $\Omega=\mathrm{LL}'$, and let A be a symmetric $n\times n$ matrix. Then the cumulants $\kappa_1, \kappa_2, \ldots$ of the distribution of $x'\mathrm{A}x$ are

(5) $$\kappa_j=2^{j-1}(j-1)!\{\mathrm{tr}(\mathrm{L}'\mathrm{AL})^j+j\mu'\mathrm{L}'^{-1}(\mathrm{L}'\mathrm{AL})^j\mathrm{L}^{-1}\mu\}.$$

COMMENT: In practice the $n\times n$ matrix L will usually be taken as (lower) triangular. But Lemma 2 (and all subsequent results) hold for any $n\times n$ matrix L satisfying $\mathrm{LL}'=\Omega$.

Proof: Let P be an orthogonal $n\times n$ matrix and Λ a diagonal $n\times n$ matrix such that

$$\mathrm{P}'\mathrm{L}'\mathrm{ALP}=\Lambda, \qquad \mathrm{P}'\mathrm{P}=\mathrm{I}_n.$$

Then we can write

$$x=\mathrm{LP}(u+\omega)$$

where $u\simeq\mathrm{N}(0,\mathrm{I}_n)$ and $\omega=\mathrm{P}'\mathrm{L}^{-1}\mu$. Hence

$$x'\mathrm{A}x=(u+\omega)'\Lambda(u+\omega)$$

and the $j-$th cumulant, as given e. g. by JOHNSON and KOTZ ([1970], Chapter 29, section 3), is

$$\begin{aligned}\kappa_j&=2^{j-1}(j-1)!(\mathrm{tr}\,\Lambda^j+j\omega'\Lambda^j\omega)\\&=2^{j-1}(j-1)!\{\mathrm{tr}(\mathrm{L}'\mathrm{AL})^j+j\mu'\mathrm{L}'^{-1}\mathrm{P}(\mathrm{P}'\mathrm{L}'\mathrm{ALP})^j\mathrm{P}'\mathrm{L}^{-1}\mu\}\\&=2^{j-1}(j-1)!\{\mathrm{tr}(\mathrm{L}'\mathrm{AL})^j+j\mu'\mathrm{L}'^{-1}(\mathrm{L}'\mathrm{AL})^j\mathrm{L}^{-1}\mu\}.\quad\square\end{aligned}$$

Combining Lemma 1 and Lemma 2, we obtain the moments of a quadratic form in normal variables in a readily usable form. [1]

LEMMA 3 : Let x be a normally distributed $n\times 1$ vector with mean μ and positive definite covariance matrix $\Omega=\mathrm{LL}'$, and let A be a symmetric $n\times n$ matrix. Then, for $s=1,2,\ldots$

(6) $$\mathrm{E}(x'\mathrm{A}x)^s=\sum_{\nu}\gamma_s(\nu)\prod_{j=1}^{s}\{\mathrm{tr}(\mathrm{L}'\mathrm{AL})^j+j\mu'\mathrm{L}'^{-1}(\mathrm{L}'\mathrm{AL})^j\mathrm{L}^{-1}\mu\}^{n_j}$$

where the summation is over all $1\times s$ vectors $\nu=(n_1,n_2,\ldots,n_s)$ whose components n_j are nonnegative integers satisfying $\sum_{j=1}^{s}jn_j=s$, and

(7) $$\gamma_s(\nu)=s!\,2^s\prod_{j=1}^{s}[n_j!(2j)^{n_j}]^{-1}.$$

1. Lemma 3 can also be obtained using zonal polynomials, see JAMES [1964], DAVIS [1979] and HILLIER and SMITH [1983].

Proof: Let $\kappa_1, \kappa_2, \ldots$ be the cumulants of the distribution of $x'Ax$, and let

$$\beta_j = \operatorname{tr}(L'AL)^j + j\mu' L'^{-1}(L'AL)^j L^{-1}\mu.$$

Then, from Lemma 2

$$(\kappa_j/j!)^{n_j} = 2^{jn_j}\beta_j^{n_j}/(2j)^{n_j}$$

and hence from Lemma 1

$$\alpha_s = s!\sum_{\nu}\prod_{j=1}^{s}[n_j!(2j)^{n_j}]^{-1}2^{jn_j}\beta_j^{n_j} = \sum_{\nu}\gamma_s(\nu)\prod_{j=1}^{s}\beta_j^{n_j}$$

since $\sum_{j=1}^{s} jn_j = s$. □

3 The Moments of a Ratio of Two Random Variables

The next ingredient needed to prove our main result concerns the moments of a ratio of two random variables and is due, in essence, to SAWA [1972]. See also MEHTA and SWAMY ([1978], Lemma 3).

LEMMA 4: Let w_1 and w_2 be two random variables such that $P(w_2 > 0) = 1$. Assume that there exists a joint moment generating function of w_1 and w_2:

$$\varphi(\theta_1, \theta_2) = E\exp(\theta_1 w_1 + \theta_2 w_2)$$

for all $|\theta_1| < \varepsilon$ and $-\infty < \theta_2 < \varepsilon$ where ε is some positive constant. Then, for $s = 1, 2, \ldots,$

$$E(w_1/w_2)^s = \frac{1}{(s-1)!}\int_0^\infty t^{s-1}\left[\frac{\partial^s}{\partial\theta^s}\varphi(\theta, -t)\right]_{\theta=0} dt, \tag{8}$$

provided the expectation exists.

Proof: For $s = 1, 2, \ldots$ we have

$$w_1^s = (\partial^s e^{\theta w_1}/\partial\theta^s)_{\theta=0} \tag{9}$$

and also

$$w_2^{-s} = \frac{1}{(s-1)!}\int_0^\infty t^{s-1}e^{-tw_2}\,dt. \tag{10}$$

$\left[\text{The latter equality, which holds for all } w_2>0, \text{ is obtained by substituting}\right.$

$x=tw_2$ in the gamma function $\left.(s-1)!=\int_0^\infty x^{s-1}e^{-x}\,dx.\right]$ Hence,

$$\text{(11)} \qquad \mathrm{E}(w_1/w_2)^s=\frac{1}{(s-1)!}\mathrm{E}\int_0^\infty t^{s-1}w_1^s\exp(-tw_2)\,dt$$

$$=\frac{1}{(s-1)!}\int_0^\infty t^{s-1}(\mathrm{E}\,w_1^s\exp(-tw_2))\,dt,$$

where the first equality follows from (10) and the fact that $\mathrm{E}(w_1/w_2)^s$ exists; the second equality is not completely trivial and follows from Fubini's theorem and an extension of it, known as Tonelli's theorem (see Dunford and Schwartz [1958], section 3.11). Also since all moments of w_1 exist it follows that $w_1^s\exp(\theta w_1)$ has a finite expectation and hence that $w_1^s\exp(\theta w_1-tw_2)$ has a finite expectation. Thus

$$\text{(12)} \qquad \mathrm{E}\,w_1^s\exp(-tw_2)=[\mathrm{E}\,w_1^s\exp(\theta w_1-tw_2)]_{\theta=0}$$

$$=\left[\mathrm{E}\frac{\partial^s}{\partial\theta^s}\exp(\theta w_1-tw_2)\right]_{\theta=0}$$

$$=\left[\frac{\partial^s}{\partial\theta^s}\varphi(\theta,-t)\right]_{\theta=0}.$$

Inserting (12) in (11) gives the desired result. □

4 The Joint Moment Generating Function of s Quadratic Forms in Normal Variables

Before we can prove our main result we need one further lemma. This concerns the joint moment generating function of two quadratic forms in normal variables. Below we present the general case with s quadratic forms.

LEMMA 5 : Let x be a normally distributed $n \times 1$ vector with mean μ and positive definite covariance matrix $\Omega = LL'$. Let $A_1, A_2, \ldots, A_s$ be s symmetric $n \times n$ matrices. Then the joint moment generating function of $x'A_1 x,\ x'A_2 x, \ldots, x'A_s x$:

$$\varphi(\theta_1, \theta_2, \ldots, \theta_s) = \mathrm{E}\exp(\theta_1 x' A_1 x + \ldots + \theta_s x' A_s x)$$

exists for all $|\theta_i| < \varepsilon$ $(i = 1, 2, \ldots, s)$ where ε is some positive constant, and is given by

$$\varphi(\theta_1, \theta_2, \ldots, \theta_s) = |I - 2C|^{-1/2} \exp\left(-\frac{1}{2}\eta'\eta\right) \exp\left(\frac{1}{2}\eta'(I - 2C)^{-1}\eta\right)$$

where

$$C = \theta_1 L' A_1 L + \ldots + \theta_s L' A_s L, \qquad \eta = L^{-1}\mu.$$

Proof: Let $u \cong N(0, I_n)$, so that $x = Lu + \mu$. Then

$$\begin{aligned}
\varphi(\theta_1, \theta_2, \ldots, \theta_s) &= \mathrm{E}\exp(u + \eta)' C (u + \eta) \\
&= (2\pi)^{-(1/2)n} \int_{\mathbb{R}^n} \exp(u + \eta)' C (u + \eta) . \exp\left(-\frac{1}{2}u'u\right) du \\
&= (2\pi)^{(1/2)n} \int_{\mathbb{R}^n} \exp - \frac{1}{2}\{(u - \alpha)'(I - 2C)(u - \alpha) - 2R\}\, du
\end{aligned}$$

where

$$\alpha = 2C(I - 2C)^{-1}\eta, \qquad R = \frac{1}{2}\eta'(I - 2C)^{-1}\eta - \frac{1}{2}\eta'\eta.$$

Hence,

$$\begin{aligned}
\varphi(\theta_1, \ldots, \theta_s) &= e^R (2\pi)^{-(1/2)n} \int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}u'(I - 2C)u\right\} du \\
&= e^R |I_n - 2C|^{-1/2},
\end{aligned}$$

provided the θ's are sufficiently close to zero to guarantee that $I_n - 2C$ is positive definite. □

5 Main Result

We now have all the ingredients to prove our main result.

THEOREM 6 : Let x be a normally distributed $n \times 1$ vector with mean μ and positive definite covariance matrix $\Omega = LL'$. Let A be a symmetric

$n \times n$ matrix and B a positive semidefinite $n \times n$ matrix, $B \neq 0$. Let P be an orthogonal $n \times n$ matrix and Λ a diagonal $n \times n$ matrix such that

$$P' L' BLP = \Lambda, \qquad P' P = I_n,$$

and define

$$A^* = P' L' ALP, \qquad \mu^* = P' L^{-1} \mu.$$

Then we have, provided the expectation exists, for $s = 1, 2, \ldots,$

$$E\left[\frac{x' A x}{x' B x}\right]^s = \frac{\exp(-(1/2)\mu' \Omega^{-1} \mu)}{(s-1)!} \sum_{\nu} \gamma_s(\nu)$$

$$\times \int_0^\infty t^{s-1} |\Delta| e^{(1/2)\xi' \xi} \prod_{j=1}^{s} (\operatorname{tr} R^j + j \xi' R^j \xi)^{n_j} \, dt$$

where the summation is over all $1 \times s$ vectors $\nu = (n_1, n_2, \ldots, n_s)$ whose elements n_j are nonnegative integers satisfying $\sum_{j=1}^{s} j n_j = s$,

$$\gamma_s(\nu) = s!\, 2^s \prod_{j=1}^{s} [n_j! (2j)^{n_j}]^{-1}$$

and Δ is a diagonal positive definite $n \times n$ matrix, R a symmetric $n \times n$ matrix and ξ an $n \times 1$ vector given by

$$\Delta = (I_n + 2t\Lambda)^{-1/2}, \qquad R = \Delta A^* \Delta, \qquad \xi = \Delta \mu^*.$$

Proof: Let $\varphi(\theta, t)$ be the joint moment generating function of $x' A x$ and $x' B x$. Then, by Lemma 5

(13) $$\varphi(\theta, -t) = |I - 2C|^{-1/2} \exp\left(-\frac{1}{2}\eta' \eta\right) \exp\left(\frac{1}{2}\eta' (I - 2C)^{-1} \eta\right)$$

where

$$C = \theta L' AL - t L' BL, \qquad \eta = L^{-1} \mu.$$

Now, with P, Δ and R as defined in the theorem, we have

$$I - 2C = I + 2t L' BL - 2\theta L' AL$$

$$= P \Delta^{-1} (I - 2\theta R) \Delta^{-1} P',$$

so that

(14) $$|I - 2C|^{-1/2} = |\Delta| . |I - 2\theta R|^{-1/2}$$

and

$$(I - 2C)^{-1} = P\Delta(I - 2\theta R)^{-1}\Delta P'. \tag{15}$$

Letting $\xi = \Delta P' L^{-1}\mu$ and inserting (14), (15) and $\eta = L^{-1}\mu$ in (13) gives

$$\begin{aligned}\varphi(\theta, -t) &= |\Delta| . |I - 2\theta R|^{-1/2} \\ &\quad \times \exp\left(-\frac{1}{2}\mu'\Omega^{-1}\mu\right)\exp\left(\frac{1}{2}\xi'(I - 2\theta R)^{-1}\xi\right) \\ &= \exp\left(-\frac{1}{2}\mu'\Omega^{-1}\mu\right)|\Delta|\exp\left(\frac{1}{2}\xi'\xi\right)\psi(\theta),\end{aligned}$$

where

$$\psi(\theta) \equiv |I - 2\theta R|^{-1/2}\exp\left(-\frac{1}{2}\xi'\xi\right)\exp\left(\frac{1}{2}\xi'(I - 2\theta R)^{-1}\xi\right).$$

Now, let w be a normally distributed $n \times 1$ vector with mean ξ and covariance matrix I_n. As a special case of Lemma 5 it appears that

$$\psi(\theta) = E\exp(\theta w' R w),$$

that is, $\psi(\theta)$ is the moment generating function of $w' R w$. Thus, using the key property of moment generating functions,

$$\left[\frac{\partial^s \psi(\theta)}{\partial\theta^s}\right]_{\theta=0} = E(w' R w)^s.$$

Hence, since $\varphi(\theta, -t)$ depends on θ only through $\psi(\theta)$, we obtain

$$\left[\frac{\partial^s \varphi(\theta, -t)}{\partial\theta^s}\right]_{\theta=0} = \exp\left(-\frac{1}{2}\mu'\Omega^{-1}\mu\right)|\Delta|\exp\left(\frac{1}{2}\xi'\xi\right)E(w' R w)^s. \tag{16}$$

Also, from Lemma 3,

$$E(w' R w)^s = \sum_{\nu}\gamma_s(\nu)\prod_{j=1}^{s}(\operatorname{tr} R^j + j\xi' R^j \xi\}^{n_j}. \tag{17}$$

Inserting (17) in (16) and using Lemma 4, the result follows. □

6 Existence of the Moments of $x' A x / x' B x$

Theorem 1 is valid if and only if the expectation of $(x' A x / x' B x)^s$ exists. In this section we establish when this is the case.

THEOREM 7 : Let x be a normally distributed $n \times 1$ vector with mean μ and positive definite covariance matrix $\Omega = LL'$. Let A be a symmetric $n \times n$ matrix and let B be a positive semidefinite $n \times n$ matrix of rank $r \geqslant 1$. If $r \leqslant n-1$, let Q be an $n \times (n-r)$ matrix of full column-rank $n-r$ such that $L' BLQ = 0$.

(i) If $r \leqslant n-1$ and $L' ALQ = 0$, or if $r = n$, then $E(x' A x / x' B x)^s$ exists for all $s \geqslant 0$.

(ii) If $r \leqslant n-1$, $Q' L' ALQ = 0$ and $L'ALQ \neq 0$, then $E(x' A x / x' B x)^s$ exists for $0 \leqslant s < r$ and does not exist for $s \geqslant r$; and

(iii) If $r \leqslant n-1$ and $Q'L'ALQ \neq 0$, then $E(x' A x / x' B x)^s$ exists for $0 \leqslant s < r/2$ and does not exist for $s \geqslant r/2$.

Proof: Assume first that $r = n$, so that B is positive definite. Then

$$\frac{x' A x}{x' B x} = \frac{(B^{1/2} x)' B^{-1/2} A B^{-1/2} (B^{1/2} x)}{(B^{1/2} x)' (B^{1/2} x)},$$

which is bounded by the extreme eigenvalues of $B^{-1/2} AB^{-1/2}$, and hence possesses moments of any order.

Next assume $1 \leqslant r \leqslant n-1$. Let P_1 be an $n \times r$ matrix satisfying

$$L' BLP_1 = P_1 \Lambda_1, \qquad P_1' P_1 = I_r, \qquad P_1' Q = 0,$$

where Λ_1 is a diagonal $r \times r$ matrix containing the r positive eigenvalues of $L'BL$. Also let $P_2 = Q(Q' Q)^{-1/2}$ and define

$$y_1 = P_1' L^{-1} x \qquad \text{and} \qquad y_2 = P_2' L^{-1} x.$$

We note that y_1 and y_2 are independently normally distributed, and that

$$P_1 P_1' + P_2 P_2' = I_n.$$

Letting

$$C_{11} = P_1' L'ALP_1, \qquad C_{12} = P_1' L'ALP_2, \qquad C_{22} = P_2' L'ALP_2.$$

we obtain

$$\frac{x' A x}{x' B x} = \frac{y_1' C_{11} y_1 + 2 y_1' C_{12} y_2 + y_2' C_{22} y_2}{y_1' \Lambda_1 y_1}. \tag{18}$$

It is clear from (18) that $E(x' A x / x' B x)^s$ exists if and only if

$$E\left[\frac{2 y_1' C_{12} y_2 + y_2' C_{22} y_2}{y_1' y_1}\right]^s \tag{19}$$

exists. Now distinguish between three cases as in the theorem.

(i) $C_{12} = 0$, $C_{22} = 0$. Then (19) obviously exists for every s.

(ii) $C_{22} = 0$, $C_{12} \neq 0$. Then (19) exists if and only if $E(y_1' C_{12} y_2 / y_1' y_1)^s$ exists. By Theorem 1 of KINAL [1980] this is the case iff $s < r$.

(iii) $C_{22} \neq 0$. Then (19) exists iff $E(y_1' y_1)^{-s}$ exists. But, since $y_1' y_1$ follows a noncentral χ^2 distribution with r degrees of freedom, $E(y_1' y_1)^{-s}$ exists iff $s < r/2$ (see JOHNSON and KOTZ [1969-1970], Chapter 26, section 2 and Chapter 30, section 3).

This completes the proof. □

7 Some Remarks on Computation

Let us define

(20) $$f(\nu, t) = t^{s-1} |\Delta| e^{(1/2)\xi'\xi} \prod_{j=1}^{s} (\operatorname{tr} R^j + j\,\xi' R^j \xi)^{n_j}$$

where Δ, R and ξ are defined in the theorem of section 5 and depend on t. The s-th moment of $x' A x / x' B x$ is then proportional to

(21) $$\sum_{\nu} \gamma_s(\nu) \int_0^{\infty} f(\nu, t)\, dt.$$

The integral in (21) has to evaluated numerically. If necessary, the entire integration range $(0, \infty)$ is first transformed to $(0, 1)$ using the identity

(22) $$\int_0^{\infty} f(\nu, t)\, dt = \int_0^1 f\left(\nu, \frac{1-z}{z}\right) \frac{1}{z^2}\, dz.$$

Instead of (21) where a number of integrals have to be evaluated numerically, we can also calculate

(23) $$\int_0^{\infty} \sum_{\nu} \gamma_s(\nu) f(\nu, t)\, dt.$$

The function under the integral sign is now more involved, but only one integral has to be evaluated. We found that (21) gives more accurate results than (23).

In addition, the expression

$$\operatorname{tr} R^j + j\xi' R^j \xi$$

can be calculated either directly or indirectly by transforming R into diagonal form:

$$S'RS = M, \qquad S'S = I_n$$

where S is orthogonal and M is diagonal. If $\mu_1, \ldots, \mu_n$ denote the diagonal elements of M and $\xi^* = S'\xi$, then

$$\operatorname{tr} R^j + j\xi' R^j \xi = \sum_{i=1}^{n} \mu_i^j (1 + j\xi_i^{*2}). \tag{24}$$

We found that, unless n is small (say $n \leqslant 20$) and s is large (say $s \geqslant 8$), the direct approach is to be preferred. It is less time consuming and also more accurate.

A major simplification in the evaluation of the first four moments in the central case was proposed by ALI [1984], section 3, who noted that the product of factors involving t can be expressed as partial fractions.

Finally, as mentioned in section 2, there is, for a given value of s, the problem: how many vectors $(n_1, n_2, \ldots, n_s)$ are there? Let us designate this number as $p(s)$. Knowledge of $p(s)$ is important, if only to know how much memory space to reserve. It is easy to see that $p(s)$ is just the number of partitions of the integer s.[2] Values of $p(s)$ are given in HALL [1967], p. 35, for $s \leqslant 100$. For $s = 1, \ldots, 12$ we find

s	1	2	3	4	5	6	7	8	9	10	11	12
$p(s)$	1	2	3	5	7	11	15	22	30	42	56	77

HALL [1967], p. 35, also provides a recurrence formula for $p(s)$ from which the values of $p(s)$ can be calculated for any value of s.

2. A partition of a positive integer s is a representation of s as a sum of positive integers $s = t_1 + t_2 + \ldots + t_k$, $t_i > 0$, $i = 1, \ldots, k$. For example, the partitions of the number 4 are (4), (3, 1), (2, 2), (2, 1, 1) and (1, 1, 1, 1), so that $p(4) = 5$.

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